Stability Criteria for Accelerating and Decelerating Aircraft

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Theme

THE linearized differential equation for the longitudinal oscillations of an aerodynamic body is derived for the general case when the atmospheric density, thrust force, and velocity are varying. New stability criteria are developed for both ascending and descending flight of a decelerating coasting vehicle having zero thrust. For constant acceleration with varying thrust in a nearly horizontal trajectory an exact solution is obtained in terms of the Coulomb wave functions.

We have used the same standard aerodynamic coefficients and flight path axes system based upon the variables (V,α,γ) (see Fig. 1) as used by Friedrich and Dore,[1] and Allen[2] to study the decelerating re-entry body. We have extended their results by obtaining exact solutions for their differential equations. We have compared our exact solution for constant acceleration along a horizontal flight path with the approximate one obtained by Oswald[3] who used the fixed principal body axes based upon the variables (u,w,q) (see Fig. 1).

Contents

This theoretical analysis was restricted to the pure longitudinal oscillations following a disturbance in the steady-state angle of attack, and was based on the same flight path axes system used by Friedrich and Dore[1] and Allen.[2] This rotating axes system is always tangent to the instantaneous flight path that is given by the steady-state straight line flight path γ_0, and the small perturbation $\gamma(t)$, both defined positive upwards as in Fig. 1. For this flight path axes system C_L is always normal to the x-axis so the nonlinear equations of motion may be written in terms of the standard aerodynamic coefficients as

$$m\dot{V} = -(1/2)\rho V^2 S C_D - W\sin(\gamma_0 + \gamma) + T\cos\alpha \quad (1)$$

$$mV\dot{\gamma} = (1/2)\rho V^2 S C_{L\alpha}\alpha - W\cos(\gamma_0 + \gamma) + T\sin\alpha \quad (2)$$

$$I(\ddot{\alpha} + \ddot{\gamma}) = (1/2)\rho V^2 SL[C_{m\alpha}\alpha + (L/V)C_{m\dot{\alpha}}\dot{\alpha} + (L/V)C_{mq}(\dot{\alpha} + \dot{\gamma})] \quad (3)$$

These equations for a planar rotating axes system are for the same type of aircraft or missile that is represented by Oswald's[3] equations for the fixed principal axes system with the exception that the thrust has been assumed to act through the c. g. so as to remain in alignment with the fixed reference axis for zero lift as shown in Fig. 1. If we now eliminate $\dot{\gamma}$ and $\ddot{\gamma}$ by introducing Eq. (2) into Eq. (3), and assume that the aerodynamic coefficients are constant, we obtain the following linearized differential equation for the variation in angle of attack (α) that is valid as long as $(gL/V^2)\sin\gamma_0 \ll 1$:

$$d^2\alpha/dt^2 + b(t)d\alpha/dt + c(t)\alpha = 0 \quad (4)$$

$$b(t) = (V/L)\mu[C_{L\alpha} - \sigma(C_{m\dot{\alpha}} + C_{mq})] + (T/mV) \quad (5)$$

$$c(t) = (V/L)^2\mu\sigma[-C_{m\alpha} - C_{mq}(\mu C_{L\alpha} + TL/mV^2)] + (V/L)\mu C_{L\alpha}(\dot{V}/V + \dot{\rho}/\rho) - (T/mV)(\dot{V}/V) + (\dot{T}/mV) \quad (6)$$

where

$$\mu = \mu(t) = \rho SL/2m; \quad \sigma = mL^2/I \quad (7)$$

and

$$(T/mV) = (\dot{V}/V) + (V/L)\mu C_D \quad (8)$$

For the case of coasting flight with zero thrust $(T = 0)$ we can reduce the preceding equations to those given by Allen[2] if we introduce the variable $Y = \beta y$, and the exponential atmospheric density variation given by Allen as $\rho = \rho_0 e^{-Y}$. Then for either a descending $(\gamma_0 < 0)$, or an ascending $(\gamma_0 > 0)$ flight path we have the change in altitude (y) given by

$$dy/dt = V\sin\gamma_0 = \beta^{-1}dY/dt \quad (9)$$

In this case, if $(gL/V^2) \ll 1$, we can write Eq. (4) in the form

$$(d^2\alpha/dY^2) + 2k_1e^{-Y}(d\alpha/dY) + (k_2e^{-Y} + k_3e^{-2Y})\alpha = 0 \quad (10)$$

The constants are the same as those of Allen,[2] namely

$$k_1 = (\delta_0/2)[C_{L\alpha} - C_D - \sigma(C_{m\dot{\alpha}} + C_{mq})] \quad (11)$$

$$k_2 = \delta_0[-\sigma C_{m\alpha}(\beta L\sin\gamma_0)^{-1} - C_{L\alpha}] \quad (12)$$

$$k_3 = \delta_0^2 C_{L\alpha}[-C_D - \sigma C_{mq}] \quad (13)$$

where

$$\delta_0 = \mu_0(\beta L\sin\gamma_0)^{-1}; \quad \mu_0 = \rho_0 SL/2m \quad (14)$$

As shown by Allen,[2] and Tobak and Allen,[4] a satisfactory solution of Eq. (10) for most cases is given by

$$\alpha(y) = [C_1J_0(\zeta^{1/2}) + C_2Y_0(\zeta^{1/2})]\exp(k_1e^{-\beta y}) \quad (15)$$

where

$$\zeta = 4(k_1 + k_2)e^{-Y} = 4(k_1 + k_2)e^{-\beta y} \quad (16)$$

and (J_0, Y_0) are the zero order Bessel functions of the first and second kind, respectively. Laitone and Vinh[5] have shown that Eq. (15) is an exact solution of Eq. (10) whenever $k_1^2 = k_3$, and it provides an excellent approximation whenever $k_2 \gg (k_1^2 - k_3) \geq 0$. As shown by Allen[2] we must take $C_2 = 0$ for the re-entry problem since ζ increases from zero to $4(k_1 + k_2)$, while on the other hand we need $C_2 \neq 0$ for ascending flight in order to satisfy the initial conditions since ζ is now decreasing to a zero value as y increases. Because $Y_0(0) \to -\infty$ we can therefore predict a critical altitude for any ascending missile if we note

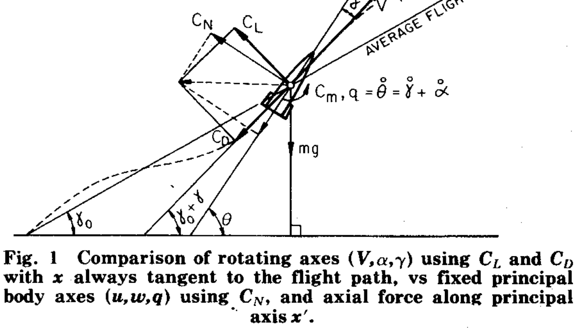

Fig. 1 Comparison of rotating axes (V,α,γ) using C_L and C_D with x always tangent to the flight path, vs fixed principal body axes (u,w,q) using C_N, and axial force along principal axis x'.

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that the zero order Bessel function $Y_0(\zeta^{1/2})$ diverges with no further oscillations whenever $\zeta^{1/2} \leq 0.8936$. Since the variable ζ decreases as y increases for an ascending missile we can determine the altitude at which α diverges with no further oscillations as

$$e^{-\beta y/2} < (0.8936/2)(k_1 + k_2)^{-1/2}; \; y > \beta^{-1}\ln[5(k_1 + k_2)] \tag{17}$$

Because δ_0 in Eq. (14) changes from positive for ascending flight ($\gamma_0 > 0$), to negative for descending flight ($\gamma_0 < 0$), we find that in the usual case $k_1 > 0$ in ascending flight and $k_1 < 0$ in descending flight. Therefore in both cases the exponential term containing k_1 in Eq. (15) produces damping. However, if C_D is excessively large then k_1 can become positive in descending flight ($\delta_0 < 0$). Then Eq. (15) predicts a critical altitude at which the oscillation amplitude starts to increase as the altitude decreases. Allen[2] derived the first approximation to this critical altitude and we have obtained the complete expression for this criterion by applying the Sonin-Pólya theorem, as given by Szegö,[6] directly to the differential equation itself, Eq. (10). The second approximation to this critical altitude for the angle of attack oscillation envelope divergence in descending flight with $k_1 > 0$ may be simplified to

$$y = \beta^{-1}\ln[4k_1 - (k_3/k_2)] \tag{18}$$

when $(k_3/k_2) = 0$ this reduces to Allen's criterion.

The angle of attack variation can be oscillatory only if $(k_1 + k_2) > 0$ because Eqs. (15) and (16) show that the oscillating Bessel functions (J_0, Y_0) are replaced by the nonoscillating Bessel functions (I_0,K_0) whenever $(k_1 + k_2) < 0$. Since $\beta^{-1} \approx 22{,}000$ ft, therefore the $C_{m\alpha}$ term in Eq. (12) completely dominates the solution as long as $L \sin \gamma_0 < 10^2$ and $-C_{m\alpha} > 10^{-3}$. Consequently in the usual case k_2 is positive and much greater in magnitude than k_1 or k_3, as shown by Allen.[2] However, if $C_{m\alpha} > 0$ then the solution is no longer oscillatory when $(k_1 + k_2) < 0$, and the aerodynamic body is then both statically and dynamically unstable. We have verified these conclusions from Eq. (15), which is Allen's approximate solution of differential Eq. (10), by applying the Sonin-Pólya theorem, as given by Szegö,[6] directly to the differential equations involved. The exact solution to Eq. (10) contains a confluent hypergeometric function and we discovered that an incorrect statement concerning its oscillatory behavior was presented in both Bateman[7] and Abramowitz and Stegun.[8] Their statement that the confluent hypergeometric function $\phi(a,1;x)$ is oscillatory for $a > 0$ and $x > ½$ is in error because in this case the Sonin-Pólya theorem is not applicable, and ϕ is actually nonoscillatory. Stone[9] used this incorrect statement to erroneously predict dynamic stability for an ascending coasting vehicle having $C_{m\alpha} > 0$.

Now let us consider an increasing thrust that maintains a constant acceleration ($\dot{V} = A$) for the case of a horizontal flight path so that $\gamma_0 = 0$, and $\rho = \rho_0$ remains constant. Then Eq. (4) may be written as

$$d^2\alpha/dt^2 + b(t)d\alpha/dt + c(t)\alpha = 0 \tag{19}$$

where

$$b(t) = (V/L)\mu_0[C_{L\alpha} + C_D - \sigma(C_{m\dot{\alpha}} + C_{mq})] + (A/V) \tag{20}$$

$$c(t) = (V/L)^2\mu_0\{-\sigma C_{m\alpha} - \mu_0\sigma C_{mq}(C_{L\alpha} + C_D) + (AL/V^2)(C_{L\alpha} + C_D - \sigma C_{mq})\} - (A/V)^2 \tag{21}$$

$$\mu_0 = (\rho_0 SL)(2m)^{-1}; \quad \sigma = mL^2/I \tag{22}$$

Since μ_0, as well as each aerodynamic coefficient, has been assumed to be constant, therefore Eq. (19) applies to a nearly horizontal trajectory with either a very low speed, or a hypersonic speed, in order to have the aerodynamic coefficients remain independent of the flight Mach number. We were able to obtain an exact solution of the above equations in terms of the Coulomb wave functions F_0 and G_0, as given by Abramowitz and Stegun,[8] for the initial conditions $\alpha(0) = \alpha_0$, $\dot{\alpha}(0) = 0$, in the form

$$\alpha(t)/\alpha_0 = (V/V_0)^{-1}e^{-[(V/V_0)^2-1]b_0/4}\{[(1 + b_0/2)(2\lambda_0)^{-1}G_0(\eta_0,\lambda_0) - G_0'(\eta_0,\lambda_0)]F_0(\eta_0,p) + [F_0'(\eta_0,\lambda_0) - (1 + b_0/2)(2\lambda_0)^{-1}F_0(\eta_0,\lambda_0)]G_0(\eta_0,p)\} \tag{23}$$

where

$$V(t) = V_0 + At; \; p(t) = \lambda_0(V/V_0)^2; \; p(0) = \lambda_0 = (1/2)(c_0 - b_0^2/4)^{1/2} > 0 \tag{24}$$

$$b_0 = \mu_0 V_0^2(AL)^{-1}[C_{L\alpha} + C_D - \sigma(C_{m\dot{\alpha}} + C_{mq})] \tag{25}$$

$$c_0 = \mu_0 V_0^4(AL)^{-2}[-\sigma C_{m\alpha} - \mu_0\sigma C_{mq}(C_{L\alpha} + C_D)] \tag{26}$$

and

$$\eta_0 = \mu_0 V_0^2(AL)^{-1}(-\sigma C_{m\dot{\alpha}})(1/4)(c_0 - b_0^2/4)^{-1/2} \tag{27}$$

A very important simplification occurs whenever $C_{m\dot{\alpha}} = 0$. In this case $\eta_0 = 0$, and the solution reduces to

$$\alpha(t) = (1 + At/V_0)^{-1} \exp\{[(1 + At/V_0)^2 - 1][-b_0/4]\} \{C_1 \sin[\lambda_0(1 + At/V_0)^2] + C_2 \cos[\lambda_0(1 + At/V_0)^2]\} \tag{28}$$

This solution simplifies to the one given by Oswald[3] if we consider $V(0) = 0$, and then replace (V/V_0) by (At/V_0); consequently Oswald's solution is only valid for the initial low speed trajectory after launching from rest. On the other hand, Eq. (28) is even valid at hypersonic speeds as long as the aerodynamic coefficients are nearly constant. Therefore in all cases having nearly constant acceleration (A) we find an exponential damping provided by $b_0 > 0$ from Eqs. (23) and (25) in the form

$$|\alpha|/\alpha_0 = (1 + At/V_0)^{-1} \exp[(1 + At/V_0)^2 - 1][-b_0/4] \tag{29}$$

For comparison with the constant velocity case we can obtain the damping directly from Eq. (19) since $V = V_0$ and $A = 0$ so that

$$|\alpha|/\alpha_0 = e^{-(b_1/2)t} \tag{30}$$

where the constant coefficient now is given by

$$b_1 = (\mu_0 V_0/L)[C_{L\alpha} + C_D - \sigma(C_{m\dot{\alpha}} + C_{mq})] \tag{31}$$

These expressions, and a similar one for the coasting case with $T = 0$, show that thrust always increases the damping, while drag decreases it.

References

[1]Friedrich, H. R. and Dore, F. J., "The Dynamic Motion of a Missile Descending Through the Atmosphere," *Journal of the Aeronautical Sciences*, Vol. 22, No. 9, Sept. 1955, pp. 628–632.

[2]Allen, H. J., "Motion of a Ballistic Missile Angularly Misaligned with the Flight Path upon Entering the Atmosphere and Its Effect upon Aerodynamic Heating, Aerodynamic Loads, and Miss Distance," TN 4048, Oct. 1957, NACA.

[3]Oswald, T. W., "Dynamic Behavior During Accelerated Flight with Particular Application to Missile Launching," *Journal of the Aeronautical Sciences*, Vol. 23, No. 8, Aug. 1956, pp. 781–791.

[4]Tobak, M. and Allen, H. J., "Dynamic Stability of Vehicles Traversing Ascending or Descending Paths through the Atmosphere," TN 4275, July 1958, NACA.

[5]Laitone, E. V. and Vinh, N. X., "Exact Solution for Dynamic Oscillations of Re-Entry Bodies," *AIAA Journal*, Vol. 10, No. 7, July 1972, pp. 925–927.

[6]Szegö, G., *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. 23, New York, 1939, p. 161.

[7]Bateman, H., *Higher Transcendental Functions*, Vol. 1, Chap. VI, McGraw-Hill, New York, 1953, p. 291.

[8]Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions*, AMS 55, National Bureau of Standards, June 1964, pp. 511, 538–542.

[9]Stone, G. W., "Aerodynamic Stability of a Coasting Vehicle Rapidly Ascending through the Atmosphere," *AIAA Journal*, Vol. 4, No. 9, Sept. 1966, pp. 1560–1565.